Advances in Mathematics **154**, 258–279 (2000)
doi:10.1006/aima.2000.1916, available online at http://www.idealibrary.com on IDEAL®

Quasidiagonal Extensions and Sequentially Trivial Asymptotic Homomorphisms

V. M. Manuilov

Department of Mechanics and Mathematics, Moscow State University, Moscow, 119899, Russia

and

K. Thomsen

Institut for Matematiske Fag, Ny Munkegade, 8000 Aarhus C, Denmark

Received December 15, 1999; accepted February 17, 2000

1. INTRODUCTION

N. Salinas defined an extension of separable C^*-algebras,

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0, \tag{1.1}$$

to be quasidiagonal when $B \otimes \mathcal{K}$ contains an approximate unit of projections which is quasi-central in E, cf. [Sa].[1] He identified, under certain conditions, the subgroup of $KK^1(A, B)$ which the quasidiagonal extensions correspond to under Kasparov's isomorphism $\mathrm{Ext}^{-1}(A, B) \simeq KK^1(A, B)$. C. Schochet has removed some of Salinas' conditions in [S], the result being that when A is a unital C^*-algebra in the bootstrap category for which the UCT holds, and there exists a unital absorbing quasidiagonal extension of A by $B \otimes \mathcal{K}$, the subgroup of the $\mathrm{Ext}^{-1}(A, B)$ corresponding to the quasidiagonal extensions can be identified with the group

$$\mathrm{Pext}(K_*(A), K_*(B)) = \mathrm{Pext}(K_0(A), K_0(B)) \oplus \mathrm{Pext}(K_1(A), K_1(B)).$$

This subgroup of $KK^1(A, B)$ plays an important role in the version of the UCT theorem which was obtained by Dadarlat and Loring [DL2], and it is very interesting to reverse the point of view and ask which extensions this subgroup of $KK^1(A, B)$ corresponds to under Kasparov's isomorphism. It is clear that despite the work of Salinas and Schochet, the answer can in general not be "the quasidiagonal extensions," because there

[1] Strictly speaking Salinas only considered essential and unital extensions, but the definition has been extended to the general case by Brown and Dadarlat in [BD].

0001-8708/00 $35.00

may not be any (e.g., when $B \otimes \mathcal{K}$ does not have an approximate unit consisting of projections). To formulate the answer which we offer in this paper, call an extension of the form (1.1) *weakly quasidiagonal* when there is an increasing sequence of projections $P_1 \leqslant P_2 \leqslant P_3 \leqslant \cdots$ in the multiplier algebra $M(E)$ of E such that

(1) $P_n E \subseteq B \otimes \mathcal{K}$, $n \in \mathbb{N}$,

(2) $\lim_{n \to \infty} P_n e - eP_n = 0$, $e \in E$,

(3) $\lim_{n \to \infty} P_n b = b$, $b \in B \otimes \mathcal{K}$.

For unital extensions (i.e., E when is unital) the two notions of quasidiagonality coincide. But contrary to quasidiagonal extensions weakly quasidiagonal extensions always exist, and our main result shows that when A is KK-equivalent to an abelian separable C^*-algebra there is a natural group isomorphism between $\operatorname{Pext}(K_*(A), K_{*-1}(B))$ and the subgroup of $\operatorname{Ext}^{-1}(SA, B)$ represented by the weakly quasidiagonal extensions of SA by $B \otimes \mathcal{K}$. Furthermore, we show without any restriction on A that the subgroup of $\operatorname{Ext}^{-1}(SA, B)$ represented by the weakly quasidiagonal extensions of SA by $B \otimes \mathcal{K}$ coincides with the subgroup represented by the quasidiagonal extensions whenever there exists a quasidiagonal extension of SA by $B \otimes \mathcal{K}$, i.e., when $B \otimes \mathcal{K}$ has an approximate unit consisting of projections. It may be, in this case, that the identification of the extensions which correspond to $\operatorname{Pext}(K_*(A), K_{*-1}(B))$ can be obtained by combining the methods of Salinas and Schochet with the result of Voiculescu that the suspension of any C^*-algebra is quasidiagonal; cf. [V, Theorem 5]. However, to handle the general case it is necessary to introduce the weakly quasidiagonal extensions.

The main tool we use to investigate extensions, and in this paper mainly the weakly quasidiagonal extensions, is the Connes–Higson construction which allows us to translate considerations about extensions to considerations about asymptotic homomorphisms. This leads us to introduce a subclass of the asymptotic homomorphisms which are important for other purposes also, namely asymptotic homomorphisms $\varphi = (\varphi_t)_{t \in [0, \infty)} : A \to B$ with the property that

$$\lim_{n \to \infty} \varphi_n(a) = 0, \qquad a \in A.$$

We call these asymptotic homomorphisms *sequentially trivial.* To get anywhere with this approach we need to be able to go back from (sequentially trivial) asymptotic homomorphisms to (weakly quasidiagonal) extensions, i.e., we need to reverse the Connes–Higson construction. This was already done in [H-LT], at the cost of an additional suspension. The key idea here is to use an alternative approach based on "discretizing" the asymptotic homomorphisms. This was suggested by the work Mishchenko and Noor Mohammad [MN] and Manuilov and Mishchenko [MM].

2. WEAKLY QUASIDIAGONAL EXTENSIONS

Throughout the paper A and B will denote separable C^*-algebras, with B stable.

An extension

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \tag{2.1}$$

of A by B is *semi-split* when there is a completely positive linear contraction $s: A \to E$ such that $p \circ s = \mathrm{Id}_A$. Two semi-split extensions,

$$0 \to B \to E_1 \to A \to 0$$

and

$$0 \to B \to E_2 \to A \to 0,$$

are homotopic when there is a semi-split extension of A by $IB = C[0, 1] \otimes B$ sitting in the middle of a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \pi_0} & & \uparrow & & \| & & \\ 0 & \longrightarrow & IB & \longrightarrow & E_3 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

where $\pi_0, \pi_1: IB \to B$ denote evaluation at 0 and 1, respectively. It follows from the work of Kasparov and others that the homotopy classes of semi-split extensions form an abelian group $\mathrm{Ext}^{-1}(A, B)$ which is isomorphic to $KK^1(A, B)$, cf. [K-JT, Theorem 3.3.14]. In the following we denote for any C^*-algebra D by $M(D)$ the multiplier algebra of D, by $Q(D)$ the generalized Calkin algebra $Q(D) = M(D)/D$ and by q_D the quotient map $q_D: M(D) \to Q(D)$.

DEFINITION 2.1. An extension (2.1) is *weakly quasidiagonal* when there is an increasing sequence $P_1 \leqslant P_2 \leqslant P_3 \leqslant \cdots$ of projections in $M(E)$ such that

(1) $P_n E \subseteq B$ for all $n \in \mathbb{N}$,

(2) $\lim_{n\to\infty} P_n e - eP_n = 0$ for all $e \in E$,

(3) $\lim_{n\to\infty} P_n b = b$ for all $b \in B$.

Following [Sa, BD] we call an extension (2.1) *quasidiagonal* when it is weakly quasidiagonal and the projections $\{P_n\}$ required by Definition 2.1 can be found in B. A crucial difference between the notion of quasidiagonal and weakly quasidiagonal extensions is that weakly quasidiagonal extensions always exist. Indeed, the direct sum extension

$$0 \to B \to B \oplus A \to A \to 0$$

is always weakly quasidiagonal, but only quasidiagonal when B has an approximate unit consisting of projections.

Let $\mathrm{Ext}_q^{-1}(A, B)$ and $Q(A, B)$ denote the sets in $\mathrm{Ext}^{-1}(A, B)$ consisting of the elements which can be represented by a weakly quasidiagonal extension, and a quasidiagonal extension, respectively. $\mathrm{Ext}_q^{-1}(A, B)$ and $Q(A, B)$ are then always abelian sub-semigroups in $\mathrm{Ext}^{-1}(A, B)$ (although $Q(A, B)$ can be empty), and $\mathrm{Ext}_q^{-1}(A, B)$ is a subgroup when A or B is a suspension. The same is true about $Q(A, B)$ if the set is not empty (as it is automatically when B is a suspension). Clearly, $Q(A, B) \subseteq \mathrm{Ext}_q^{-1}(A, B)$.

From the work of Busby we know that the extensions of A by B can be identified with $\mathrm{Hom}(A, Q(B))$. This will be done systematically in the following and we need therefore the following characterization of the weakly quasidiagonal extensions in terms of their Busby invariant. The simple proof is left to the reader.

LEMMA 2.2. *Let $\varphi \in \mathrm{Hom}(A, Q(B))$, and let $\lambda: A \to M(B)$ be a map such that $\varphi(a) = q_B \circ \lambda(a)$, $a \in A$. Then φ is weakly quasidiagonal if and only if there is a sequence $P_1 \leqslant P_2 \leqslant P_3 \leqslant \cdots$ of projections in $M(B)$ such that*

(a) $P_n \lambda(A) \subseteq B$, $n \in \mathbb{N}$

(b) $\lim_{n\to\infty} P_n \lambda(a) - \lambda(a) P_n = 0$ *for all* $a \in A$,

(c) $\lim_{n\to\infty} P_n b = b$, $b \in B$.

LEMMA 2.3. *Let $\varphi: A \to A_1$ and $\psi: B \to B_1$ be *-homomorphisms. Assume that B and B_1 are stable. Then*

$$\varphi^*(\mathrm{Ext}_q^{-1}(A_1, B)) \subseteq \mathrm{Ext}_q^{-1}(A, B),$$

and

$$\psi_*(\mathrm{Ext}_q^{-1}(A, B)) \subseteq \mathrm{Ext}_q^{-1}(A, B_1).$$

Proof. The assertion about φ^* is trivial. To prove the assertion about ψ, recall from [T1] that ψ is homotopic to a quasi-unital *-homomorphism. So by homotopy invariance we may assume that ψ is quasi-unital, i.e., that there is a projection $e \in M(B_1)$ such that $\overline{\psi(B)\, B_1} = eB_1$. Let $\lambda: A \to M(B)$ be a completely positive map such that $q_B \circ \lambda \in \mathrm{Hom}(A, Q(B))$. Let $\{P_n\}$ be a sequence of projections in $M(B)$ meeting the conditions of Lemma 2.2. Then

$$\psi_*[q_B \circ \lambda] = [q_{B_1} \circ \bar{\psi} \circ \lambda],$$

where $\bar{\psi}: M(B) \to M(B_1)$ is the unique extension of ψ which is strictly continuous on the unit ball, cf. [T1]. Note that $\bar{\psi}(1) = e$. Set $Q_n = \bar{\psi}(P_n) + (1-e)$. It is straightforward to check that this is a sequence of projections in $M(B_1)$ satisfying the three conditions of Lemma 2.2 relative to $\bar{\psi} \circ \lambda$. ∎

For any C^*-algebra D we let $s_D: D \to \mathcal{K} \otimes D$ be the *-homomorphism given by $s_D(d) = e \otimes d$ for some minimal non-zero projection e in $\mathcal{K}$.

Corollary 2.4.

$$s_{B*} : \mathrm{Ext}_q^{-1}(A, B) \to \mathrm{Ext}_q^{-1}(A, \mathcal{K} \otimes B)$$

and

$$s_A^* : \mathrm{Ext}_q^{-1}(\mathcal{K} \otimes A, B) \to \mathrm{Ext}_q^{-1}(A, B)$$

are both isomorphisms.

Proof. The inverses of the isomorphisms $s_{B*} : \mathrm{Ext}^{-1}(A, B) \to \mathrm{Ext}^{-1}(A, \mathcal{K} \otimes B)$ and $s_A^* : \mathrm{Ext}^{-1}(\mathcal{K} \otimes A, B) \to \mathrm{Ext}^{-1}(A, B)$ are given by

$$[\varphi] \mapsto \mu_*[\varphi], \qquad \varphi \in \mathrm{Hom}(A, Q(\mathcal{K} \otimes B)),$$

and

$$[\psi] \mapsto \mu_*[\mathrm{Id}_{\mathcal{K}} \otimes \psi], \qquad \psi \in \mathrm{Hom}(A, Q(B)),$$

respectively, where $\mu: \mathcal{K} \otimes B \to B$ is a *-isomorphism. So they both respect weak quasidiagonality. ∎

Proposition 2.5. *Let*

$$0 \to B \to E \to A \to 0$$

be a weakly quasidiagonal extension. By applying the K-functor we obtain a pure extension

$$0 \to K_*(B) \to K_*(E) \to K_*(A) \to 0.$$

Proof. The proof of [BD, Theorem 8] can be taken over ad verbatim. ∎

For a convenient short introduction to pure extensions, see [BD, Sect. 2] or [R], where the notion was first introduced in the classification program.

It follows from Proposition 2.5 that there is a map

$$\tau : \mathrm{Ext}_q^{-1}(A, B) \to \mathrm{Pext}(K_*(A), K_*(B)),$$

where

$$\mathrm{Pext}(K_*(A), K_*(B)) = \mathrm{Pext}(K_0(A), K_0(B)) \oplus \mathrm{Pext}(K_1(A), K_1(B)).$$

τ is natural with respect to the functoriality in both A and B.

3. FROM EXTENSIONS TO ASYMPTOTIC HOMOMORPHISMS

In [CH] Connes and Higson used a procedure which out of a given extension of the form (2.1) produces an asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [0, \infty)} : SA \to B$. In [H-LT] it was shown that by applying their construction to a semi-split extension the resulting asymptotic homomorphism is asymptotically equal to a completely positive asymptotic homomorphism, i.e., an asymptotic homomorphism, $\varphi = (\varphi_t)_{t \in [0, \infty)}$, where the individual maps, the φ_t's, are all completely positive linear contractions. By using universal properties of the functors involved, it was concluded in [H-LT] that

$$KK(A, B) \simeq [[SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]]_{cp},$$

where $[[\cdot, \cdot]]_{cp}$ denotes the homotopy classes of completely positive asymptotic homomorphisms. We need to know that this isomorphism is actually effectuated by the Connes–Higson construction. For this we need the following.

THEOREM 3.1 (Dadarlat and Loring [DL1]). *Let B be a stable C^*-algebra. The suspension map*

$$S : [[SA, B]]_{cp} \to [[S^2A, SB]]_{cp}$$

is an isomorphism.

Proof. This was proved in [DL1] for E-theory, $[[\cdot, \cdot]]$. It is straightforward to check that their argument works equally well with completely positive asymptotic homomorphisms.

THEOREM 3.2. *Let B be a stable C^*-algebra. Then the Connes–Higson construction gives rise to an isomorphism*

$$CH : \operatorname{Ext}^{-1}(A, B) \to [[SA, B]]_{cp}.$$

Proof. From [H-LT, Sect. 4] we take the commuting diagram

$$\begin{tikzcd} \operatorname{Ext}^{-1}(A, B) \arrow[r, "S"] \arrow[d, "CH"'] & \operatorname{Ext}^{-1}(SA, SB) \arrow[d, "CH"] \\ {[[SA, B]]_{cp}} \arrow[r, "-S"] \arrow[ur, "M"] & {[[S^2A, SB]]_{cp}} \end{tikzcd} \tag{3.1}$$

From the work of Kasparov we know that $S : \operatorname{Ext}^{-1}(A, B) \to \operatorname{Ext}^{-1}(SA, SB)$ is an isomorphism; cf. [Sk, 6.7]. Since the lower suspension map is an isomorphism by Theorem 3.1, it follows that CH: $\operatorname{Ext}^{-1}(A, B) \to [[SA, B]]_{cp}$ is an isomorphism. ∎

We are going to identify the image of $\operatorname{Ext}_q^{-1}(A, B)$ under the CH-map when A is a suspension. The first step in this direction is the following.

LEMMA 3.3. *Let*

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \tag{3.2}$$

be a weakly quasidiagonal semi-split extension. There is then a completely positive asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [0, \infty)} : SA \to B$, which is homotopic to the asymptotic homomorphism obtained by applying the Connes–Higson construction to (3.2) *and satisfies that*

$$\lim_{n \to \infty} \varphi_n(a) = 0, \qquad a \in SA.$$

Proof. Let $\{P_n\}$ be the sequence of projections in $M(E)$ meeting the conditions of Definition 2.1, and let s: $A \to E$ be a set-theoretic right-inverse for p. By convex interpolation we can construct a norm-continuous path

V_t, $t \in [0, \infty)$, of elements $0 \leqslant V_t \leqslant 1$ in $M(E)$ such that $V_t E \subseteq B$, $t \in [0, \infty)$, $\lim_{t \to \infty} V_t e - e V_t = 0$ for all $e \in E$, $\lim_{t \to \infty} V_t b = b$, $b \in B$, and $V_n = P_n$ for all $n \in \mathbb{N}$. By using the same arguments as in [CH] we obtain an asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [0, \infty)} : SA \to B$ such that

$$\lim_{t \to \infty} \|\varphi_t(f \otimes a) - f(V_t)\, s(a)\| = 0$$

for all $f \in S = C_0(0, 1)$, $a \in A$. To see that φ is homotopic to the asymptotic homomorphism coming from a genuine Connes–Higson construction, consider an approximate unit $u_t \in B$, $t \in [0, \infty)$, which asymptotically commutes with E. A convex combination of u_t and V_t can then be used to obtain a homotopy between φ and the result of the proper Connes–Higson construction where one uses $\{u_t\}$. Note also that

$$f(V_n)\, s(a) = f(P_n)\, s(a) = 0$$

for all $n \in \mathbb{N}$, $f \in S$, $a \in A$. It follows readily that $\lim_{n \to \infty} \varphi_n(x) = 0$ for all $x \in SA$. Since the extension (3.2) is semi-split by assumption, we may assume that s is a completely positive contraction. By (the proof of) Lemma 4.1 of [H-LT], there is a completely positive asymptotic homomorphism $\sigma = (\sigma_t)_{t \in [0, \infty)}: SA \to B$ such that $\lim_{t \to \infty} \|\sigma_t(x) - \varphi_t(x)\| = 0$ for all $x \in SA$. ∎

4. SEQUENTIALLY TRIVIAL ASYMPTOTIC HOMOMORPHISMS

Motivated by Lemma 3.3 we make the following definition.

Definition 4.1. An asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [0, \infty)}: A \to B$ is said to be sequentially trivial when

$$\lim_{n \to \infty} \varphi_n(a) = 0$$

for all $a \in A$.

In this paper we shall only be concerned with sequentially trivial asymptotic homomorphisms which are also completely positive. We denote by $[[A, B]]^0_{cp}$ the elements of $[[A, B]]_{cp}$ which can be represented by a sequentially trivial completely positive asymptotic homomorphism.

The composition product of Connes and Higson gives us a product

$$[[A, B]]_{cp} \times [[B, C]]_{cp} \ni (x, y) \mapsto y \bullet x \in [[A, C]]_{cp}.$$

The construction of this product and the proof of its associativity is much simpler in the present case where we deal with asymptotic homomorphisms which are also completely positive. So we will only summarize the construction here.

A *parametrization* $r: [0, \infty) \to [0, \infty)$ is a continuous function such that $\lim_{t \to \infty} r(t) = \infty$. Given two parametrizations, r and s, we write $r \preccurlyeq s$ when $r(t) \leqslant s(t)$ for all large enough t.

THEOREM 4.2. *There is a map*

$$[[A, B]]_{cp} \times [[B, C]]_{cp} \ni (x, y) \mapsto y \bullet x \in [[A, C]]_{cp}$$

with the following properties:

(a) (*Definition*) *When $\varphi: A \to B$ and $\psi: B \to C$ are completely positive asymptotic homomorphisms there is a parametrization $r: [0, \infty) \to [0, \infty)$ such that $\lambda = \{\psi_{s(t)} \circ \varphi_t\}_{t \in [0, \infty)}$ is a completely positive asymptotic homomorphism for every parametrization s with $r \preccurlyeq s$, and*

$$[\psi] \bullet [\varphi] = [\lambda].$$

(b) (*Associativity*)

$$z \bullet (y \bullet x) = (z \bullet y) \bullet x.$$

The map $\bullet$ will be called *the composition product.*

LEMMA 4.3. (a) *Let $x \in [[A, B]]^0_{cp}$, $y \in [[B, C]]_{cp}$. Then*

$$y \bullet x \in [[A, C]]^0_{cp}.$$

(b) *Let $x \in [[A, B]]_{cp}$, $y \in [[B, C]]^0_{cp}$. Then*

$$y \bullet x \in [[A, C]]^0_{cp}.$$

Proof. (a) Let $\varphi = (\varphi_t)_{t \in [0, \infty)} : A \to B$ and $\psi = (\psi_t)_{t \in [0, \infty)} : B \to C$ be completely positive asymptotic homomorphisms representing x and y, respectively. We may assume that φ is sequentially trivial. For any parametrization $r: [0, \infty) \to [0, \infty)$ we have that

$$\|\psi_{r(n)} \circ \varphi_n(a)\| \leqslant \|\varphi_n(a)\|.$$

Hence $\lim_{n \to \infty} \psi_{r(n)} \circ \varphi_n(a) = 0$ since $\lim_{n \to \infty} \varphi_n(a) = 0$.

(b) Let $\varphi = (\varphi_t)_{t \in [0, \infty)} : A \to B$ and $\psi = (\psi_t)_{t \in [0, \infty)} : B \to C$ be completely positive asymptotic homomorphisms representing x and y, respectively. We may assume that ψ is sequentially trivial. Let $r: [0, \infty) \to [0, \infty)$

be a parametrization such that $(\psi_{s(t)} \circ \varphi_t)_{t \in [0, \infty)}$ represents $y \bullet x$ for any parametrization $r \leqslant s$. Let $\{a_1, a_2, a_3, \cdots\}$ be a dense sequence in A. Since ψ is sequentially trivial we can find an increasing sequence $m_1 < m_2 < m_3 < \cdots$ of natural numbers such that $m_n \geqslant \sup_{t \in [0, n+1]} r(t)$ and

$$\|\psi_k(\varphi_n(a_i))\| \leqslant \frac{1}{n}, \qquad i = 1, 2, \cdots, n, k \geqslant m_n.$$

Let s: $[0, \infty) \to [0, \infty)$ be a parametrization which is linear on $[n-1, n]$ and has $s(n) = m_n$ for all $n \in \mathbb{N}$. Then $(\psi_{s(t)} \circ \varphi_t)_{t \in [0, \infty)}$ represents $y \bullet x$ (since $r \leqslant s$) and is sequentially trivial. ∎

As a very special case of Lemma 4.3 we see that we can make $[[A, B]]^0_{cp}$ functorial in both A and B by restricting the functoriality of $[[A, B]]_{cp}$. It follows also that we have isomorphisms

$$s_{A^*} : [[\mathscr{K} \otimes SA, B]]^0_{cp} \to [[SA, B]]^0_{cp}$$

and

$$s_{B_*} : [[SA, B]]^0_{cp} \to [[SA, \mathscr{K} \otimes B]]^0_{cp},$$

when B is stable.

In addition we get the following:

LEMMA 4.4. *Let B be a stable C^*-algebra. The suspension map $S : [[SA, B]]^0_{cp} \to [[S^2A, SB]]^0_{cp}$ is an isomorphism.*

Proof. By using the isomorphisms s_{A^*} and s_{SA^*} from above, we may assume that A is stable. We then see from [DL1] that the inverse of the isomorphism $S : [[SA, B]]_{cp} \to [[S^2A, SB]]_{cp}$ of Theorem 3.1 is given by

$$x \mapsto [\alpha] \bullet (Sx) \bullet [\beta],$$

where $Sx \in [[S^3A, S^2B]]_{cp}$ is the suspension of $x \in [[S^2A, SB]]_{cp}$, $\beta \in \mathrm{Hom}(SA, S^3A)$ and α: $S^2B \to B$ is a completely positive asymptotic homomorphism. By Lemma 4.3, this inverse map takes $[[S^2A, SB]]^0_{cp}$ into $[[SA, B]]^0_{cp}$. ∎

5. FROM ASYMPTOTIC HOMOMORPHISMS TO EXTENSIONS.

LEMMA 5.1. *Let $\{\psi_t\}_{t \in [0, \infty)}$: $A \to B$ be a completely positive asymptotic homomorphism. There is then a sequence $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots$ in $[0, \infty)$ such that*

(a) $\lim_{n\to\infty} t_n = \infty$.

(b) $\lim_{n\to\infty} \sup_{s\in[t_n, t_{n+1}]} \|\psi_{t_n}(a) - \psi_s(a)\| = 0$ *for all* $a \in A$.

Proof. Let $a_1, a_2, a_3, ...$ be a dense sequence in A. Let $n \in \mathbb{N}$. There is a $\delta_n > 0$ such that

$$\|\psi_s(a_i) - \psi_t(a_i)\| \leqslant \frac{1}{n}$$

for all $s, t \in [n-1, n]$ with $|s-t| < \delta_n$ and all $i = 1, 2, ..., n$. Choose $m_n \in \mathbb{N}$ so large that $1/m_n < \delta_n$, and set

$$s_i^n = n - 1 + \frac{i}{m_n}, \qquad i = 0, 1, 2, ..., m_n.$$

Let $t_0 < t_1 < t_2 < \cdots$ be the sequence given by reading the list

$$\begin{array}{cccccc} s_0^1, & s_1^1, & s_2^1, & s_3^1, & ..., & s_{m_1}^1 \\ s_0^2, & s_1^2, & s_2^2, & s_3^2, & ..., & s_{m_2}^2 \\ s_0^3, & s_1^3, & s_2^3, & s_3^3, & ..., & s_{m_3}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Then $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \sup_{s\in[t_n, t_{n+1}]} \|\psi_{t_n}(a_i) - \psi_s(a_i)\| = 0$ for all i. Since $\{a_i\}$ is dense in A and the ψ_s's are uniformly bounded (b) follows immediately. ∎

The sequence of maps $\varphi_{t_n}: A \to B$, $n = 0, 1, 2, 3, ...$, from Lemma 5.1 will be called a *discretization* of $\{\varphi\}_{t\in[0,\infty)}$.

Let $H_B = l_2(\mathbb{Z}) \otimes B$ be the Hilbert B-module consisting of bi-infinite sequences $(... b_{-3}, b_{-2}, b_{-1}, b_0, b_1, b_2, b_3, ...)$ in B such that $\sum_{i=-\infty}^{\infty} b_i^* b_i$ converges in B. We identify $M(\mathscr{K} \otimes B)$ and $\mathscr{K} \otimes B$ with the adjoinable and the compact operators on H_B, respectively. For any given norm-bounded sequence $(x_i)_{i=0}^{\infty}$ in $M(B)$ we can define an element

$$D[(x_i)] \in M(\mathscr{K} \otimes B)$$

by

$$D[(x_i)](b_j)_k = \begin{cases} x_k b_k, & k \geqslant 0, \\ 0, & k < 0, \end{cases} \qquad (b_j) \in H_B.$$

Note that $D[(x_i)] \in \mathscr{K} \otimes B$ if and only if $x_i \in B$ for all i, and $\lim_{i\to\infty} \|x_i\| = 0$.

Let $\varphi_{t_n}: A \to B$, $n = 0, 1, 2, 3, \cdots$, be a discretization of the completely positive asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [0, \infty)}: A \to B$. We can then define a map $\mathbf{\Phi}^0_{(t_i)}: A \to M(\mathcal{K} \otimes B)$ by

$$\mathbf{\Phi}^0_{(t_i)}(a) = D[\varphi_{t_i}(a)].$$

It is easy to see that $\mathbf{\Phi}^0_{(t_n)}$ is a *-homomorphism modulo $\mathcal{K} \otimes B$, and we denote the resulting extension of A by $\mathcal{K} \otimes B$ by $\mathbf{\Phi}_{(t_n)} \in \operatorname{Hom}(A, Q(\mathcal{K} \otimes B))$. The two-sided shift

$$V_0(b_j) = (b_{j+1}),\ (b_j) \in H_B,$$

is a unitary in $M(\mathcal{K} \otimes B)$, and we have that

$$V_0 \mathbf{\Phi}^0_{(t_n)}(a) - \mathbf{\Phi}^0_{(t_n)}(a)\, V_0 \in \mathcal{K} \otimes B$$

for all $a \in A$ because of condition (b) in Lemma 5.1. It follows that

$$V \mathbf{\Phi}_{(t_n)}(a) = \mathbf{\Phi}_{(t_n)}(a)\, V \tag{5.1}$$

in $Q(\mathcal{K} \otimes B)$ for all $a \in A$, where V denotes the image of V_0 in $Q(\mathcal{K} \otimes B)$. It follows from (5.1) and the fact that V is unitary, that we may define an extension $\mathbf{\Psi}_{(t_n)} \in \operatorname{Hom}(C(\mathbb{T}) \otimes A, Q(\mathcal{K} \otimes B))$ such that

$$\mathbf{\Psi}_{(t_n)}(f \otimes a) = f(V)\, \mathbf{\Phi}_{(t_n)}(a), \qquad f \in C(\mathbb{T}),\ a \in A.$$

Lemma 5.2. $\mathbf{\Psi}_{(t_n)} \in Ext^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B)$.

Proof. By [A, Theorem 6] (combined with [K-JT, Lemma 3.2.8]) it suffices to find a sequence $\chi_m: C(\mathbb{T}) \otimes A \to M(\mathcal{K} \otimes B)$ of completely positive contractions such that

$$\lim_{m \to \infty} q_{\mathcal{K} \otimes B} \circ \chi_m(f) = \mathbf{\Psi}_{(t_n)}(f), \qquad f \in C(\mathbb{T}) \otimes A. \tag{5.2}$$

For each $m \in \mathbb{N}$ we choose a partition of unity $\{h^m_i : i = 1, ..., m\}$ in $C(\mathbb{T})$ such that the maximal diameter of the support of the h^m_i 's tend to 0 as m tends to infinity. For each m, i, take an element x^m_i in the support of h^m_i. Define $\chi_m: C(\mathbb{T}) \otimes A \to M(\mathcal{K} \otimes B)$ by

$$\chi_m(f) = \sum_{i=1}^{m} \sqrt{h^m_i}(V_0) \mathbf{\Phi}^0_{(t_n)}(f(x^m_i)) \sqrt{h^m_i}(V_0).$$

Then χ_m is a completely positive contraction. For every $g \in C(\mathbb{T})$ and $a \in A$,

$$q_{\mathcal{K} \otimes B} \circ \chi_m(g \otimes a) = \left[\sum_{i=1}^{m} g(x^m_i)\, h^m_i \right] (V)\, \mathbf{\Phi}_{(t_n)}(a).$$

Hence

$$\lim_{m\to\infty} \|q_{\mathscr{K}\otimes B} \circ \chi_m(g \otimes a) - \mathbf{\Psi}_{(t_n)}(g \otimes a)\| = 0$$

since $\lim_{m\to\infty} \|\sum_{i=1}^{m} g(x_i^m)\, h_i^m - g\| = 0$ in $C(\mathbb{T})$. Since all maps in sight are linear contractions, (5.2) follows. ∎

LEMMA 5.3. *Up to homotopy* (*by semi-split extensions*) $\mathbf{\Psi}_{(t_n)}$ *is independent of the chosen discretization.*

Proof. Let $\{\varphi_{s_n}\}$ be a discretization of $\{\varphi_{t_n}\}$ with the property that $\{s_n\}$ is a refinement of $\{t_n\}$, meaning that $\{t_n\} \subseteq \{s_n\}$ (counting multiplicity). It is not difficult to see that two given discretizations have a common refinement in this sense, so to prove the lemma it suffices to prove that $\mathbf{\Psi}_{(s_n)}$ is homotopic to $\mathbf{\Psi}_{(t_n)}$.

Let $x_1^0 \leqslant x_2^0 \leqslant \cdots \leqslant x_{m_0}^0$ be the elements from $\{s_n\}$ which lie between the first occurrence of t_0 and the last occurrence of t_1. For $n \geqslant 1$, let $x_1^n \leqslant x_2^n \leqslant \cdots \leqslant x_{m_n}^n$ be the elements from $\{s_n\}$ which lie between the last occurrence of t_n and the last occurrence of t_{n+1}. For each $t \in [0, 1]$, each n and each $i = 1, 2, ..., m_n$, set

$$y_t^{n,i} = t x_i^n + (1-t)\, t_{n+1}.$$

Let s_n^t, $n = 0, 1, 2, ...$, denote the sequence obtained by reading the list

$$\begin{array}{llll} t_0, & y_t^{0,1}, & y_t^{0,2}, ..., & y_t^{0,m_0}, \\ t_1, & y_t^{1,1}, & y_t^{1,2}, ..., & y_t^{1,m_1}, \\ t_2, & y_t^{2,1}, & y_t^{2,2}, ..., & y_t^{2,m_2}, \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Then $[0,1] \ni t \mapsto \mathbf{\Phi}^0_{(s_n^t)}$ is path of completely positive contractions such that $t \mapsto \mathbf{\Phi}^0_{(s_n^t)}(a)$ is normcontinuous for all $a \in A$, $\mathbf{\Phi}^0_{(s_n^t)}(a)\, \mathbf{\Phi}^0_{(s_n^t)}(b) - \mathbf{\Phi}^0_{(s_n^t)}(ab) \in \mathscr{K} \otimes B$ for all $a, b \in A$, $\mathbf{\Phi}^0_{(s_n^0)} = \mathbf{\Phi}^0_{(s_n)}$ and $\mathbf{\Phi}^0_{(s_n^1)} = \mathbf{\Phi}^0_{(t'_n)}$, where $\{t'_n\}$ is the sequence

$$t_0, \underbrace{t_1, t_1, ..., t_1}_{m_0 \text{ repetitions}}, t_1, \underbrace{t_2, t_2, ..., t_2}_{m_1 \text{ repetitions}}, t_2, \underbrace{t_3, t_3, ..., t_3}_{m_2 \text{ repetitions}}, t_3, t_4, t_4, ...$$

By using convex combinations of t_1 and t_2, we construct in the same way a path $[0,1] \ni t \mapsto \mathbf{\Phi}^0_{(s_n^{1,t})}$ of completely positive contractions such that

$t \mapsto \mathbf{\Phi}^0_{(s^1_n, t)}(a)$ is norm-continuous for all $a \in A$, $\mathbf{\Phi}^0_{(s^1_n, t)}(a)\,\mathbf{\Phi}^0_{(s^1_n, t)}(b) - \mathbf{\Phi}^0_{(s^1_n, t)}(ab) \in \mathcal{K} \otimes B$ for all $a, b \in A$, $\mathbf{\Phi}^0_{(s^1_n, 0)} = \mathbf{\Phi}^0_{(t'_n)}$ and $\mathbf{\Phi}^0_{(s^1_n, 1)} = \mathbf{\Phi}^0_{(t''_n)}$, where $\{t''_n\}$ is the sequence

$$t_0, t_1, \underbrace{t_2, t_2, t_2, ..., t_2}_{m_0+m_1+1 \text{ repetitions}}, \underbrace{t_3, t_3, ..., t_3}_{m_2 \text{ repetitions}}, t_3, \underbrace{t_4, t_4, ..., t_4}_{m_3 \text{ repetitions}}, t_4, t_5, ...$$

By proceeding in this way, using convex combinations of t_2 and t_3, and then t_3 and t_4, etc., we get a path $[0, 1[\ \ni t \mapsto \mathbf{\Psi}_t$, of completely positive contractions such that $t \mapsto \mathbf{\Psi}_t(a)$ is normcontinuous for all $a \in A$, $\mathbf{\Psi}_t(a)\,\mathbf{\Psi}_t(b) - \mathbf{\Psi}_t(ab) \in \mathcal{K} \otimes B$ for all $a, b \in A$, $\mathbf{\Psi}_0 = \mathbf{\Phi}^0_{(s_n)}$ and $\mathbf{\Psi}_{1-1/k} = \mathbf{\Phi}^0_{(w^k_n)}$, $k \geqslant 3$, where $\{w^k_n\}$ is the sequence

$$t_0, t_1, ..., t_{k-2}, \underbrace{t_{k-1}, t_{k-1}, ..., t_{k-1}}_{m_0+m_1+\cdots+m_{k-2}+1 \text{ repetitions}}, \underbrace{t_k, t_k, ..., t_k}_{m_k \text{ repetitions}}, t_k,$$
$$\underbrace{t_{k+1}, t_{k+1}, ..., t_{k+1}}_{m_{k+1} \text{ repetitions}}, t_{k+1}, ...$$

Set $\mathbf{\Psi}_1 = \mathbf{\Phi}^0_{(t_n)}$. For each $a \in A$, define $\mathbf{\Psi}(a) \in M(I\mathcal{K} \otimes B)$ by

$$(\mathbf{\Psi}(a)\, f)(t) = \mathbf{\Psi}_t(a)\, f(t),\ t \in [0, 1], f \in I\mathcal{K} \otimes B.$$

It is easy to see that $q_{I\mathcal{K} \otimes B} \circ \mathbf{\Psi}$ is a $*$-homomorphism. If T_0 denotes the canonical image of V_0 in $M(I\mathcal{K} \otimes B)$, we have that

$$\mathbf{\Psi}(a)\, T_0 - T_0 \mathbf{\Psi}(a) \in I\mathcal{K} \otimes B$$

for all $a \in A$. This follows from the construction of $\mathbf{\Psi}$ and condition (b) of Lemma 5.1. Hence $\mathbf{\Psi}$ gives rise to an extension of $C(\mathbb{T}) \otimes A$ by $I\mathcal{K} \otimes B$, and the argument from the proof of Lemma 5.2 shows that this extension is semi-split. It is then clear that it gives a homotopy connecting $\mathbf{\Psi}_{(s_n)}$ to $\mathbf{\Psi}_{(t_n)}$. ∎

By Lemma 5.3, $[\mathbf{\Psi}_{(t_n)}] \in \operatorname{Ext}^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B)$ is independent of the sequence $\{t_n\}$ we choose to discretize $\{\varphi_t\}$ with, and consequently we denote this element of the extension group by $\mathbf{\Psi}_\varphi$.

LEMMA 5.4. *Let $\{\varphi^1_t\}_{t \in [0, \infty)}$, $\{\varphi^2_t\}_{t \in [0, \infty)} : A \to B$ be completely positive asymptotic homomorphism which are homotopic. It follows that $\mathbf{\Psi}_{\varphi^1} = \mathbf{\Psi}_{\varphi^2}$ in* $\operatorname{Ext}^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B)$.

Proof. Let $\{\psi_t\}_{t \in [0, \infty)} : A \to IB$ be a completely positive asymptotic homomorphism given rise to a homotopy between $\{\varphi^1_t\}$ and $\{\varphi^2_t\}$. Let

$\{\psi_{t_n}\}$ be a discretization of $\{\psi_t\}$. Since a refinement of the sequence $\{t_n\}$ will still be a discretization we may assume that $\{\varphi^1_{t_n}\}$ and $\{\varphi^2_{t_n}\}$ are discretizations of $\{\varphi^1_t\}$ and $\{\varphi^2_t\}$, respectively. It is then obvious that $\mathbf{\Psi}_\psi \in \mathrm{Ext}^{-1}(C(\mathbb{T}) \otimes A, I\mathcal{K} \otimes B)$ defines a homotopy connecting $\mathbf{\Psi}_{\varphi^1}$ and $\mathbf{\Psi}_{\varphi^2}$. ∎

It follows that we get a well-defined map

$$E_0\colon [[A, B]]_{cp} \to \mathrm{Ext}^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B),$$

given by $E_0[\varphi] = \mathbf{\Psi}_\varphi$. There is an obvious map $\mathrm{Ext}^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B) \to \mathrm{Ext}^{-1}(SA, \mathcal{K} \otimes B)$ induced by the embedding $S \subset C(\mathbb{T})$, and by composing with E_0 we get a map

$$E\colon [[A, B]]_{cp} \to \mathrm{Ext}^{-1}(SA, \mathcal{K} \otimes B).$$

To compare the map E with other general constructions we remind the reader of the existence of an asymptotic homomorphism

$$\chi = \{\chi_t\}_{t \in [0, \infty)} : SC(\mathbb{T}) \to \mathcal{K}$$

coming via the Connes–Higson construction from the Toeplitz extension:

$$0 \longrightarrow \mathcal{K} \longrightarrow T \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

To describe it, choose a sequence of continuous functions $\kappa_n\colon [0, \infty) \to [0, 1]$, $n = 0, 1, 2, 3, ...,$ such that

$$\lim_{t \to \infty} \kappa_n(t) = 1, \qquad n \in \mathbb{N}, \tag{5.3}$$

$$\lim_{n \to \infty} \kappa_n(t) = 0, \qquad t \in [0, \infty), \tag{5.4}$$

and

$$\lim_{n \to \infty} \sup_{t \in [0, \infty)} \|\kappa_{n+1}(t) - \kappa_n(t)\| = 0. \tag{5.5}$$

One way of constructing such a sequence of functions is to set $a_n = \sum_{i=1}^n \frac{1}{i}$ and let κ_n, $n \geqslant 1$, be the function

$$\kappa_n(t) = \begin{cases} 0, & t \in [0, a_n] \\ t - a_n, & t \in [a_n, a_n + 1] \\ 1, & t \geqslant a_n + 1. \end{cases}$$

But the actual choice is not important. Let $s\colon C(\mathbb{T}) \to M(\mathcal{K})$ be a completely positive map such that $q_{\mathcal{K}} \circ s(f) = f(T)$ for all $f \in C(\mathbb{T})$, where $T \in Q(\mathcal{K})$ is the image of the one-sided shift on $l^2(\mathbb{N})$. There is then a completely positive asymptotic homomorphism, $\chi = (\chi_t)_{t \in [0, \infty)}\colon SC(\mathbb{T}) \to \mathcal{K}$ such that

$$\lim_{t \to \infty} \|\chi_t(f \otimes g) - diag(f(\kappa_1(t)), f(\kappa_2(t)), f(\kappa_3(t)), ...)\, s(g)\| = 0$$

for all $f \in S$, $g \in C(\mathbb{T})$; cf. [H-LT, Lemma 4.1]. (Note that we are viewing $\mathcal{K}$ as the compact operators on $l_2(\mathbb{N})$, rather than $l_2(\mathbb{Z})$, as above.) By restricting χ to $S^2 \subset SC(\mathbb{T})$, we get a completely positive asymptotic homomorphism $\chi_0 : S^2 \to \mathcal{K}$ which gives rise to Bott-periodicity in E-theory.

It is important to observe that we can replace the sequence $\kappa_1, \kappa_2, \kappa_3, \cdots$ in the definition of χ and χ_0 by any other sequence $\kappa'_1, \kappa'_2, \kappa'_3, \cdots$ of continuous functions which satisfy (5.3), (5.4), and (5.5). Such a different choice will not change the classes of χ and χ_0 in $[[SC(\mathbb{T}), \mathcal{K}]]_{cp}$ and $[[S^2, \mathcal{K}]]_{cp}$, respectively.

Lemma 5.5. *The diagram*

$$\begin{tikzcd}
\mathrm{Ext}^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B) \arrow[dr, "CH"] & \\
{[[A, B]]_{cp}} \arrow[u, "E_0"] \arrow[r, "{[\varphi] \mapsto [\chi \otimes \varphi]}"'] & {[[SC(\mathbb{T}) \otimes A, \mathcal{K} \otimes B]]_{cp}}
\end{tikzcd}$$

commutes.

Proof. Let $\varphi = \{\varphi_t\}_{t \in [0, \infty)}\colon A \to B$ be a completely positive asymptotic homomorphism. For any discretization $\{\varphi_{t_n}\}$ of $\{\varphi_t\}_{t \in [0, \infty)}$, and any sequence of functions (κ_n) such that (5.3), (5.4), and (5.5) hold, set $\varphi_{t_n} = \kappa_n = 0$ for negative integers n. Let z^j, $j \geqslant 0$, denote the polynomial $z \mapsto z^j$. $CH \circ E_0[\varphi] \in [[SC(\mathbb{T}) \otimes A, \mathcal{K} \otimes B]]_{cp}$ is then represented by an asymptotic homomorphism ψ such that

$$\lim_{t \to \infty} \|\psi_t(f \otimes z^j \otimes a) - D[f(\kappa_i(t))\, \varphi_{t_i}(a)]\, V_0^j\| = 0,$$

for all $f \in S$, $a \in A$ and $j \in \mathbb{N}$. On the other hand $[\chi \otimes \varphi]$ is represented by an asymptotic homomorphism λ such that

$$\lim_{t \to \infty} \|\lambda_t(f \otimes z^j \otimes a) - D[f(\kappa_i(t))\, \varphi_t(a)](PV_0)^j)\| = 0,$$

for all $f \in S$, $a \in A$ and $j \geqslant 0$. P denotes here the orthogonal projection P: $l_2(\mathbb{Z}) \otimes B \to l_2(\mathbb{N}) \otimes B$. Note that

$$\lim_{t \to \infty} \|D[f(\kappa_i(t))\, \varphi_{t_i}(a)]\, V_0^j - D[f(\kappa_i(t))\, \varphi_{t_i}(a)](PV_0)^j\|$$
$$= \lim_{t \to \infty} \|D[f(\kappa_i(t))\, \varphi_{t_i}(a)]\, PV_0^j - D[f(\kappa_i(t))\, \varphi_{t_i}(a)](PV_0)^j\| = 0.$$

Hence

$$\lim_{t \to \infty} \|\psi_t(f \otimes z^j \otimes a) - D[f(\kappa_i(t))\, \varphi_{t_i}(a)](PV_0)^j\| = 0$$

for all f, a and j. By using the freedom in the choice of the κ_i's we can arrange that there is a sequence $0 < m_1 < m_2 < \cdots$ in $\mathbb{N}$ such that

$$\kappa_i(t) = 1, \qquad t \in [t_j, t_{j+1}], \quad i = 1, 2, ..., m_j \tag{5.6}$$

and

$$\kappa_i(t) = 0, \qquad t \in [t_j, t_{j+1}], \quad i \geqslant m_{j+1}. \tag{5.7}$$

Define a new sequence $s_1 \leqslant s_2 \leqslant s_3 \cdots$ in $[0, \infty)$ such that

$$s_i = 0, \qquad 0 \leqslant i < m_1,$$
$$s_{m_1} = s_{m_1+1} = \cdots = s_{m_2-1} = t_1,$$
$$s_{m_2} = s_{m_2+1} = \cdots = s_{m_3-1} = t_2,$$
$$s_{m_3} = s_{m_3+1} = \cdots = s_{m_4-1} = t_3,$$

and so on. Then $\{\varphi_{s_n}\}$ is also a discretization of φ, and we may therefore assume that

$$\lim_{t \to \infty} \|\psi_t(f \otimes z^j \otimes a) - D[f(\kappa_i(t))\, \varphi_{s_i}(a)](PV_0)^j\| = 0$$

for all $f \in S$, $a \in A$ and $j \in \mathbb{N}$. After these changes we have that

$$\lim_{t \to \infty} \|\psi_t(f \otimes z^j \otimes a) - \lambda_t(f \otimes z^j \otimes a)\|$$
$$\leqslant \lim_{t \to \infty} \sup_n \|f(\kappa_n(t))\, \varphi_{s_n}(a) - f(\kappa_n(t))\, \varphi_t(a)\|$$
$$= 0,$$

when $f \in S$, $a \in A$ and $j \in \mathbb{N}$. Since elements of the form $f \otimes z^j \otimes a$ generate $SC(\mathbb{T}) \otimes A$ as a C^*-algebra, it follows that $\lim_{t \to \infty} \|\psi_t(z) - \lambda_t(z)\| = 0$ for all $z \in SC(\mathbb{T}) \otimes A$. Hence $[\psi] = [\lambda]$ in $[[SC(\mathbb{T}) \otimes A, \mathcal{K} \otimes B]]_{cp}$. ▮

It follows of course that also the diagram

$$\begin{tikzcd} \mathrm{Ext}^{-1}(SA, \mathcal{K} \otimes B) \arrow[dr, "CH"] & \\ {[[A, B]]_{cp}} \arrow[u, "E"] \arrow[r, "{[\varphi] \mapsto [\chi_0 \otimes \varphi]}"'] & {[[S^2 A, \mathcal{K} \otimes B]]_{cp}} \end{tikzcd} \tag{5.8}$$

commutes, and it is this diagram we shall actually use in the following.

Lemma 5.6. $E_0([[A, B]]^0_{cp}) \subseteq \mathrm{Ext}_q^{-1}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B)$.

Proof. Let $\varphi = (\varphi_t)_{t \in [0, \infty)} : A \to B$ be a sequentially trivial completely positive asymptotic homomorphism. It is easy to see that there is a completely positive asymptotic homomorphism $(\psi_t)_{t \in [0, \infty)}$ such that $\psi_n = 0$ for all $n \in \mathbb{N}$ and $\lim_{t \to \infty} \|\psi_t(a) - \varphi_t(a)\| = 0$ for all $a \in A$. We may therefore assume that $\varphi_n = 0$ for all $n \in \mathbb{N}$. We can then find a discretization (φ_{t_n}) of φ such that there is an increasing sequence $m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ with the property that $m_{i+1} > m_i + i$ and

$$t_j = t_{m_i + 1} \in \mathbb{N}, \qquad j = m_i + 1, m_i + 2, m_i + 3, ..., m_i + i,$$

for all $i \in \mathbb{N}$. For each $k \in \mathbb{N}$, define a sequence $(p^k_i)_{i=0}^\infty$ in $M(B)$ by

$$p^k_i = \begin{cases} 1, & 0 \leqslant i \leqslant k \\ 0, & i > k. \end{cases}$$

Let $F: l_2(\mathbb{Z}) \otimes B \to l_2(\mathbb{Z}) \otimes B$ be the projection

$$F(b_j)_k = \begin{cases} b_k, & k < 0, \\ 0, & k \geqslant 0, \end{cases} \qquad (b_j) \in H_B.$$

Then $P_k = F + D[(p^k_i)]$ is a projection in $M(\mathcal{K} \otimes B)$ for all $k \in \mathbb{N}$. Note that

$$P_{m_k} \mathbf{\Phi}^0_{(t_n)}(a)\, V_0^j = \mathbf{\Phi}^0_{(t_n)}(a)\, V_0^j\, P_{m_k} \tag{5.9}$$

for all $a \in A$ and all $-k < j < k$. It follows that

$$\lim_{k \to \infty} \|P_{m_k} z - z P_{m_k}\| = 0$$

for all $z \in q^{-1}_{\mathcal{K}\otimes B}(\mathbf{\Psi}_{(t_n)}(C(\mathbb{T})\otimes A))$. Since we clearly have that $P_{m_i} \leqslant P_{m_{i+1}}$, $\lim_{k\to\infty} P_{m_k} b = b$ for all $b \in B$, and $P_{m_i} z \in \mathcal{K}\otimes B$ for all $i \in \mathbb{N}$ and all $z \in q^{-1}_{\mathcal{K}\otimes B}(\mathbf{\Psi}_{(t_n)}(C(\mathbb{T})\otimes A))$, it follows that $\mathbf{\Psi}_\varphi \in \mathrm{Ext}_q^{-1}(C(\mathbb{T})\otimes A, \mathcal{K}\otimes B)$. ∎

THEOREM 5.7. *Assume that B is stable. The maps $s_{B_*^{-1}} \circ E_0$: $[[SA, B]]^0_{cp} \to \mathrm{Ext}_q^{-1}(S^2A, B)$ and CH: $\mathrm{Ext}_q^{-1}(S^2A, B) \to [[S^3A, B]]^0_{cp}$ are isomorphisms.*

Proof. By using the stabilizing isomorphisms, $\mathrm{Ext}_q^{-1}(\mathcal{K}\otimes A, B) \to \mathrm{Ext}_q^{-1}(A, B)$, $[[S\mathcal{K}\otimes A, B]]^0_{cp} \to [[SA, B]]^0_{cp}$ and $[[S^3\mathcal{K}\otimes A, B]]^0_{cp} \to [[S^3A, B]]^0_{cp}$, we may assume that A is stable. By combining the diagram (5.8) with (3.1) we get the following commuting diagram

$$\begin{tikzcd}
 & \mathrm{Ext}_q^{-1}(S^2A, \mathcal{K}\otimes B) \arrow[r, "S"] \arrow[d, "CH"] & \mathrm{Ext}^{-1}(S^3A, S(\mathcal{K}\otimes B)) \\
{[[SA, B]]^0_{cp}} \arrow[ur, "E_0"] \arrow[r, "\lambda"'] & {[[S^3A, \mathcal{K}\otimes B]]^0_{cp},} \arrow[ur, "M"] &
\end{tikzcd} \tag{5.10}$$

where $\lambda[\varphi] = [\chi_0 \otimes \varphi]$. Since A is stable the inverse of λ: $[[SA, B]]_{cp} \to [[S^3A, \mathcal{K}\otimes B]]_{cp}$ is given by $[\varphi] \to [\psi_1 \circ \varphi \circ \psi_2]$, where ψ_1: $\mathcal{K}\otimes B \to B$ and ψ_2: $SA \to S^3A$ are both $*$-homomorphisms. This inverse clearly takes $[[S^3A, \mathcal{K}\otimes B]]^0_{cp}$ into $[[SA, B]]^0_{cp}$, so we see that the λ of diagram (5.10) is an isomorphism. Since the suspension map S in the diagram is injective (being the restriction to $\mathrm{Ext}_q^{-1}(S^2A, \mathcal{K}\otimes B)$ of an isomorphism), we conclude from the diagram that both the CH-map and the E_0-map are isomorphisms. ∎

COROLLARY 5.8. *Assume that B is stable and has an approximate unit consisting of projections. Then $Q(S^2A, B) = \mathrm{Ext}_q^{-1}(S^2A, B)$.*

Proof. Let $q_1 \leqslant q_2 \leqslant q_3 \leqslant \cdots$ be an approximate unit for B consisting of projections. If we let $n_1 < n_2 < n_3 < \cdots$ be a sufficiently rapidly increasing sequence, and we if use the sequence

$$p_i^k = \begin{cases} q_{n_k}, & 0 \leqslant i \leqslant k \\ 0, & i > k \end{cases}$$

in the proof of Lemma 5.6, we see that $E_0([[SA, B]]^0_{cp}) \subseteq Q(S^2A, \mathcal{K}\otimes B)$. It follows then from Theorem 5.7 that $Q(S^2A, \mathcal{K}\otimes B) = \mathrm{Ext}_q^{-1}(S^2A, \mathcal{K}\otimes B)$. Since B is stable, $\mathcal{K}\otimes B \simeq B$, and the conclusion follows. ∎

THEOREM 5.9. *Assume that B is stable. The map $CH : \mathrm{Ext}_q^{-1}(SA, B) \to [[S^2A, B]]_{cp}^0$ is an isomorphism. If B has an approximate unit consisting of projections we have in addition that $Q(SA, B) = \mathrm{Ext}_q^{-1}(SA, B)$.*

Proof. We may assume that A is stable. There is then a *-homomorphism $\psi: SA \to S^3A$ which is invertible in KK. There is a commuting diagram

$$\begin{array}{ccc} \mathrm{Ext}_q^{-1}(S^3A, B) & \xrightarrow{\psi^*} & \mathrm{Ext}_q^{-1}(SA, B) \\ {\scriptstyle CH}\Big\downarrow & & \Big\downarrow{\scriptstyle CH} \\ [[S^4A, B]]_{cp}^0 & \xrightarrow{\psi^*} & [[S^2A, B]]_{cp}^0 . \end{array}$$

By Theorem 5.7 the CH-map to the left is an isomorphism, and it follows from Theorem 3.1 that $\psi^* : [[S^4A, B]]_{cp} \to [[S^2A, B]]_{cp}$ is an isomorphism. The inverse is obtained by taking the composition product with an element of $[[S^4A, S^2A]]_{cp}$, so it follows from Lemma 4.3 that also the last ψ^* in the diagram is an isomorphism. Therefore the right CH-map is surjective. By Theorem 3.2 this map is also injective, and hence an isomorphism. Since $\psi^*(Q(S^3A, B)) \subseteq Q(SA, B)$, and $Q(S^3A, B) = \mathrm{Ext}_q^{-1}(S^3A, B)$ when B has an approximate unit consisting of projections, by Corollary 5.8, it follows that $Q(SA, B) = \mathrm{Ext}_q^{-1}(SA, B)$ in this case. ∎

6. PUTTING THE PIECES TOGETHER

Let A be a C^*-algebra in $\mathcal{N}$-the class of C^*-algebras for which the UCT holds (with any B). Then the map

$$\tau : \mathrm{Ext}_q^{-1}(A, B) \to \mathrm{Pext}(K_*(A), K_*(B))$$

is injective. Indeed, when $\delta : \mathrm{Pext}(K_*(A), K_*(B)) \to \mathrm{Ext}^{-1}(A, B)$ is the injection from the UCT we have that $\delta \circ \tau$ is the inclusion $\mathrm{Ext}_q^{-1}(A, B) \subseteq \mathrm{Ext}^{-1}(A, B)$.

LEMMA 6.1. *Let A be a unital abelian C^*-algebra, and assume that B has an approximate unit consisting of projections. Then*

$$\tau : \mathrm{Ext}_q^{-1}(A, B) \to \mathrm{Pext}(K_*(A), K_*(B))$$

and

$$\tau : \mathrm{Ext}_q^{-1}(SA, SB) \to \mathrm{Pext}(K_*(SA), K_*(SB)) \simeq \mathrm{Pext}(K_*(A), K_*(B))$$

are isomorphisms.

Proof. Being abelian A is automatically quasidiagonal relative to B. It follows therefore from Theorem 1.4 of [S] that already the restriction of τ to $Q(A, B)$ is surjective. Since the suspension of extensions gives us a commuting diagram

$$\begin{tikzcd} \mathrm{Ext}_q^{-1}(A, B) \arrow[r, "\tau"] \arrow[d, "s"] & \mathrm{Ext}(K_*(A), K_*(B)) \arrow[d, equal] \\ \mathrm{Ext}_q^{-1}(SA, SB) \arrow[r, "\tau"] & \mathrm{Pext}(K_*(SA), K_*(SB)), \end{tikzcd}$$

where we now know that the upper τ-map is an isomorphism, and the lower τ-map is injective, it follows that the lower τ-map is also an isomorphism. ∎

THEOREM 6.2. *Assume that A is KK-equivalent to a separable abelian C^*-algebra. It follows that*

$$\mathrm{Ext}_q^{-1}(SA, \mathscr{K} \otimes B) \simeq [[S^2A, \mathscr{K} \otimes B]]^0_{cp} \simeq \mathrm{Pext}(K_*(A), K_{*-1}(B)).$$

The isomorphisms are natural in A and B.

Proof. By combining Lemma 6.1 with Theorem 5.9 we see that there is an isomorphism

$$[[SA, \mathscr{K} \otimes B]]^0_{cp} \simeq [[S^2A, S\mathscr{K} \otimes B]]^0_{cp} \simeq \mathrm{Ext}_q^{-1}(SA, S\mathscr{K} \otimes B)$$
$$\simeq \mathrm{Pext}(K_*(A), K_*(B)),$$

when A is a unital abelian C^*-algebra and B has an approximate unit consisting of projections. The combined map, $[[SA, \mathscr{K} \otimes B]]^0_{cp} \to \mathrm{Pext}(K_*(A), K_*(B))$, can be defined for arbitrary (separable) C^*-algebras, A and B, and is natural in both variables, not only with respect to $*$-homomorphisms but in fact with respect to the pairings with KK-theory. The last assertion may be deduced from [T2, Theorem 4.9]. Hence, by adjoining units and using the split-exactness of all functors involved, we see first that the map is an isomorphism for any (separable) abelian A and any (separable) B, and then that it is also an isomorphism for any separable B, and any separable A which is KK-equivalent to a separable abelian C^*-algebra. So for any such A and B we have that

$$\mathrm{Ext}_q^{-1}(SA, \mathscr{K} \otimes B) \simeq [[S^2A, \mathscr{K} \otimes B]]_{cp}^0 \simeq \mathrm{Pext}(K_*(A), K_{*-1}(B))$$

by Theorem 5.9. ∎

REFERENCES

[A] W. Arveson, Notes on extensions of C^*-algebras, *Duke Math. J.* **44** (1977), 329–355.

[BD] L. G. Brown and M. Dadarlat, Extensions of C^*-algebras and quasidiagonality, *J. London Math. Soc. (2)* **53** (1996), 582–600.

[CH] A. Connes and N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante, *C.R. Acad. Sci. Paris Sér. I Math.* **313** (1990), 101–106.

[DL1] M. Dadarlat and T. Loring, K-homology, asymptotic representations and unsuspended E-theorie, *J. Funct. Anal.* **126** (1994), 367–383.

[DL2] M. Dadarlat and T. Loring, A universal multicoefficient theorem for the Kasparov groups, *Duke Math. J.* **84** (1996), 355–377.

[H-LT] T. Houghton-Larsen and K. Thomsen, Universal (co)homology theories, *K-Theory* **16** (1999), 1–27.

[K-JT] K. Knudsen-Jensen and K. Thomsen, "Elements of KK-Theory," Birkhäuser, Boston, 1991.

[MM] V. M. Manuilov and A. S. Mishchenko, Asymptotic and Fredholm representations of discrete groups, *Mat. Sb.* **189**, No. 10 (1998), 53–72 [In Russian]; English translation, *Russian Acad. Sci. Sb. Math.*, in press.

[MN] A. S. Mishchenko and Noor Mohammad, Asymptotic representations of discrete groups, *in* "Lie Groups and Lie Algebras: Their Representations, Generalizations and Applications," Mathematics and Its Applications, Vol. 433, Kluwer Academic, Dordrecht, 1998.

[R] M. Rørdam, Classification of certain infinite simple C^*-algebras, *J. Funct. Anal.* **131** (1995), 415–458.

[Sa] N. Salinas, Relative quasidiagonality and KK-theory, *Houston J. Math.* **18** (1992), 97–116.

[S] C. L. Schochet, The fine structure of the Kasparov groups. II. Relative quasidiagonality, preprint, 1998.

[Sk] G. Skandalis, Kasparov's bivariant K-theory and applications, *Exposition. Math.* **9** (1991), 193–250.

[T1] K. Thomsen, Homotopy classes of $*$-homomorphisms between stable C^*-algebras and their multiplier algebra, *Duke Math. J.* **61** (1990), 67–104.

[T2] K. Thomsen, Asymptotic homomorphisms and equivariant KK-theory, *J. Funct. Anal.* **163** (1999), 324–343.

[V] D. Voiculescu, A note on quasidiagonal C^*-algebras and homotopy, *Duke Math. J.* **62** (1991), 267–271.